General Solution to Linear Differential Equations

A linear differential equation can be written in the form:

$$\frac{dy}{dx} = a(x)y + b(x)$$

If b(x) = 0, the equation is homogenous. If b(x) is not zero, it is non-homogenous. The solution to a homogenous differential equation can be solved by separating variables and integrating. A non-homogenous equation can be solved similarly with an extra step.

The general solution to a linear differential equation is given by:

$$y(x) = y_h(x) + y_p(x)$$

Where $y_h(x)$ is the solution to the homogenous equation and $y_p(x)$ is the particular solution of the non-homogenous equation. We can find the particular solution by guessing its form, then plugging it back in for y. We can use b(x) to guess which form the particular solution takes.

For example, take the equation $\frac{dy}{dx} = 2y + e^{3x}$. Separation of variables lets us find the homogenous solution $y_h(x) = Ce^{2x}$,where C is some constant. We then use b(x) = e^{3x} and guess the form for the particular solution to be $y_p(x) = ae^{\beta x}$. Now, plugging this back in for y gets us:

$$3Ae^{3x} = 2(Ae^{3x}) + e^{3x}$$

We want to find some constant A, so by doing some simple algebra we figure out that A = 1. Finally, we can declare our particular solution as $y_p(x) = e^{3x}$, giving us a general solution of

$$y(x) = Ce^{2x} + e^{3x}$$

Particular Solution forms	
b(x)	$y_p(x)$
$ae^{\beta x}$	$Ae^{\beta x}$
$a\cos(\beta x)$	$A\cos(\beta x) + Bsin(\beta x)$
$b\sin(\beta x)$	$A\cos(\beta x) + Bsin(\beta x)$
$a\cos(\beta x) + bsin(\beta x)$	$A\cos(\beta x) + Bsin(\beta x)$
n^{th} degree polynomial	$A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$